

# Subdominant pseudoultrametric on graphs

O. Dovgoshey and E. Petrov

## Abstract

Let  $(G, w)$  be a weighted graph. The necessary and sufficient conditions under which a weight  $w : E(G) \rightarrow \mathbb{R}^+$  can be extended to a pseudoultrametric on  $V(G)$  are found. A criterion of the uniqueness of this extension is also obtained. It is proved that  $G$  is complete  $k$ -partite with  $k \geq 2$  if and only if, for every pseudoultrametrizable weight  $w$ , there exists the smallest pseudoultrametric agreed with  $w$ . We characterize the structure of graphs for which the subdominant pseudoultrametric is an ultrametric for every strictly positive pseudoultrametrizable weight.

**Key words:** weighted graph, infinite graph, ultrametric space, shortest path metric, complete  $k$ -partite graph.

**2010 AMS Classification:** 05C10, 05C12, 54E35.

## 1 Introduction

Throughout this paper, a *graph* is a pair  $(V, E)$  consisting of nonempty set  $V$  and (probably empty) set  $E$  elements of which are unordered pairs of different points from  $V$ . For the graph  $G = (V, E)$ , the set  $V = V(G)$  and  $E = E(G)$  are called *the set of vertices* and, respectively, *the set of edges*. Generally we shall follow terminology adopted in [1]. Let us give some definitions. If  $V(H) \subseteq V(G)$  and  $E(H) \subseteq E(G)$ , then graph  $H$  is a *subgraph* of graph  $G$ ,  $H \subseteq G$ . Recall that  $G$  is called *complete* if every two different vertices  $u, v$  are *adjacent*,  $\{u, v\} \in E(G)$ . Graph  $G$  is *finite* if  $|V(G)| < \infty$ . If  $E(G) = \emptyset$ , then  $G$  is an *empty graph*. A finite nonempty graph  $P \subseteq G$  is a *path* (in  $G$ ), if we can enumerate without repetition vertices from  $P$  into a sequence  $(v_1, v_2, \dots, v_n)$  such that

$$(\{v_i, v_j\} \in E(P)) \Leftrightarrow (|i - j| = 1).$$

We shall identify the path  $P$  with the sequence  $(v_1, v_2, \dots, v_n)$  and shall say that  $P$  connects  $v_1$  and  $v_n$ . A finite graph  $C$  is a *cycle* if  $|V(C)| \geq 3$  and there exists an enumeration  $(v_1, v_2, \dots, v_n)$  of his vertices such that

$$(\{v_i, v_j\} \in E(C)) \Leftrightarrow (|i - j| = 1 \quad \text{or} \quad |i - j| = n - 1).$$

Some two vertices in graph are *connected* if there exists a path connecting them. A graph is *connected* if every two his vertices are connected. A graph  $G = (V, E)$  together with function  $w : E \rightarrow \mathbb{R}^+ = [0, +\infty)$  is called a *weighted graph*, and  $w$  is called a *weight* or a *weighting function*. The weighted graphs we shall denote by  $(G, w)$ .

Recall now some necessary definitions from the theory of metric spaces. An *ultrametric* on a set  $X$  is a function  $d : X \times X \rightarrow \mathbb{R}^+$  such that for all  $x, y, z \in X$ :

- (i)  $d(x, y) = d(y, x)$ ,
- (ii)  $(d(x, y) = 0) \Leftrightarrow (x = y)$ ,
- (iii)  $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ .

If (ii) is replaced by the weaker condition (ii')  $d(x, x) = 0$ , then  $d$  is a *pseudoultrametric*. Inequality (iii) is often called the *strong triangle inequality*. A function  $d : X \times X \rightarrow \mathbb{R}^+$  satisfying the ordinary triangle inequality and having properties (i)-(ii'), is called a *pseudometric*.

If  $(G, w)$  is a weighted graph and

$$2 \max_{e \in E(C)} w(e) \leq \sum_{e \in E(C)} w(e) \quad (1.1)$$

for every cycle  $C \subseteq G$ , then there exists a pseudometric  $d : V(G) \times V(G) \rightarrow \mathbb{R}^+$  such that

$$w(\{x, y\}) = d(x, y) \quad (1.2)$$

for every  $\{x, y\} \in E(G)$ . As an example of such pseudometric, for connected  $G$ , we can take the well known “shortest path metric”. This result was proved in [2] and the next question was formulated. Under what conditions on  $w$  there exists an ultrametric (pseudoultrametric)  $d$  extending the weight  $w$ , in the sense that (1.2) holds for all edges  $\{x, y\}$  of  $G$ ?

Theorem 3.3 below gives us a complete answer on this question. The necessary and sufficient conditions of uniqueness of such extension are found in Theorem 5.7. Moreover, for connected  $G$ , we find the “greatest” pseudoultrametric  $d$ , extending  $w$ , and we show, that this pseudoultrametric is subdominant for the “shortest path metric” (see Theorem 3.7 and corollary 2.8). The necessary and sufficient conditions under which the subdominant pseudoultrametric is a metric are found in Theorem 4.4. Using this theorem in Corollary 4.6 we find the structural characteristic of graphs  $G$ , for which there exists  $w : E(G) \rightarrow \mathbb{R}^+$  such that:

- (i)  $w(e) > 0$  for all  $e \in E(G)$ ;
- (ii) The set of pseudoultrametrics, extending  $w$ , is not empty, but does not contain any ultrametric.

Moreover, we give some results, showing that the subdominant pseudoultrametric and the shortest path metric “behave similarly”.

## 2 Subdominant pseudoultrametric

In the next lemma and further we identify a pseudoultrametric space  $(X, d)$  with the complete weighted graph  $(G, w_d)$  having  $V(G) = X$  and satisfying the equality

$$w_d(\{x, y\}) = d(x, y) \quad (2.1)$$

for every pair of different points  $x, y \in X$ .

**Lemma 2.1.** *Let  $(X, d)$  be a pseudoultrametric space. Then for every cycle  $C \subseteq G(X)$  there exist at least two distinct edges  $e_1, e_2$  such that*

$$w_d(e_1) = w_d(e_2) = \max_{e \in E(C)} w_d(e). \quad (2.2)$$

*Proof.* Let us denote by  $q(C)$  the number of edges of a cycle  $C$ . If  $q(C) = 3$ , then (2.2) follows from the strong triangle inequality. Suppose that (2.2) holds when  $q(C) \leq n$ , but there exists a cycle  $C$  with  $q(C) = n + 1$ , having exactly one edge  $e_1 = \{x, y\}$  such that

$$w_d(e_1) = \max_{e \in C} w_d(e).$$

Let  $z$  be a vertex of cycle  $C$  adjacent to  $y$  and distinct from  $x$ . By the uniqueness of the edge of maximal weight we have

$$d(y, z) < d(x, y).$$

This inequality and the strong triangle inequality imply  $d(x, z) = d(x, y)$ . Let  $C_1$  be a cycle for which

$$V(C_1) = V(C) \setminus \{y\} \quad \text{и} \quad E(C_1) = (E(C) \setminus \{\{x, y\}, \{y, z\}\}) \cup \{\{x, z\}\}.$$

Then  $q(C_1) = n$  and  $\{x, z\}$  is the unique edge of maximal weight, which contradicts the induction hypothesis.  $\square$

**Remark 2.2.** Probably, this lemma is known. In any case, the presence in the graph of two edges of maximum length is a commonly meeting phenomenon under the work with the ultrametrics and their generalizations. For example, the so called 2-ultrametric spaces are characterized by the fact that every their four-point subspace has at least two edges with the length equals to the diameter of the subspace (see [3]).

We turn now to the definition of the subdominant pseudoultrametric.

On the set  $\mathfrak{F}$  of the pseudometrics defined on  $X$  we introduce the partial order  $\preceq$  as

$$(d_1 \preceq d_2) \Leftrightarrow (\forall x, y \in X : d_1(x, y) \leq d_2(x, y)). \quad (2.3)$$

In [4], for given metric space  $(X, d)$ , the subdominant ultrametric is defined as the greatest element of the poset  $(\mathfrak{F}_d, \preceq)$ , where  $\mathfrak{F}_d \subseteq \mathfrak{F}$  is the set of the ultrametrics  $\delta$  such that

$$\delta(x, y) \leq d(x, y)$$

for  $x, y \in X$ . We generalize this definition to the weighted graphs.

**Definition 2.3.** Let  $(G, w)$  be a nonempty weighted graph and  $\mathfrak{F}_{w,u}$  be the family of the pseudoultrametrics  $\rho$  such that

$$\rho(u, v) \leq w(\{u, v\})$$

for every edge  $\{u, v\} \in E(G)$ . If the poset  $(\mathfrak{F}_{w,u}, \preceq)$  contains the greatest element, then we call this element the subdominant pseudoultrametric for  $w$ .

Note that  $\mathfrak{F}_{w,u} \neq \emptyset$  because the zero pseudoultrametric

$$\rho(u, v) = 0, \forall u, v \in V(G)$$

belongs to  $\mathfrak{F}_{w,u}$ .

We turn now to the construction of subdominant pseudoultrametrics.

Let  $u, v$  be two distinct vertices of a connected weighted graph  $(G, w)$ . Denote by  $\mathfrak{P}_{u,v}$  the set of the paths connecting  $u$  and  $v$ . Define the function  $\rho_w$  on the Cartesian square  $V(G) \times V(G)$  by the rule

$$\rho_w(x, y) := \begin{cases} 0 & \text{if } x = y \\ \inf_{P \in \mathfrak{P}_{x,y}} (\max_{e \in P} w(e)) & \text{if } x \neq y. \end{cases} \quad (2.4)$$

**Theorem 2.4.** The function  $\rho_w$  is the subdominant pseudoultrametric for every nonempty connected weighted graph  $(G, w)$ .

*Proof.* Let us verify the strong triangle inequality

$$\rho_w(u, v) \leq \max\{\rho_w(u, p), \rho_w(p, v)\} \quad (2.5)$$

for different vertices  $u, v, p \in V(G)$ . Let  $\varepsilon$  be an arbitrary positive number. There exist some paths  $P_1 \in \mathfrak{P}_{u,p}$  and  $P_2 \in \mathfrak{P}_{p,v}$  such that

$$\rho_w(u, p) + \varepsilon \geq \max_{e \in P_1} w(e) \quad \text{и} \quad \rho_w(p, v) + \varepsilon \geq \max_{e \in P_2} w(e). \quad (2.6)$$

The subgraph  $G_1$  of  $G$  with  $V(G_1) = V(P_1) \cup V(P_2)$  and  $E(G_1) = E(P_1) \cup E(P_2)$  is connected. Let  $P_3$  be a path in  $G_1$ , connecting  $u$  and  $v$ . Then using (2.6) we find

$$\rho_w(u, v) \leq \max_{e \in P_3} w(e) \leq (\max_{e \in P_1} w(e)) \vee (\max_{e \in P_2} w(e)) \leq \max\{\rho_w(u, p) + \varepsilon, \rho_w(p, v) + \varepsilon\}.$$

Hence, letting  $\varepsilon$  to zero, we obtain (2.5).

It remains to verify that  $\rho_w$  is subdominant. Suppose there exist  $\rho \in \mathfrak{F}_{w,u}$  and  $v_1, v_2 \in V(G)$  such that

$$\rho(v_1, v_2) > \rho_w(v_1, v_2). \quad (2.7)$$

This inequality and (2.4) imply the existence of a path  $P \in \mathfrak{P}_{v_1, v_2}$  for which

$$\rho(v_1, v_2) > \max_{e \in P} w(e).$$

Note that  $\rho(u, v) \leq w(\{u, v\})$  for every  $\{u, v\} \in E(G)$ . Consequently, the path  $P$  does not contain  $\{v_1, v_2\}$ . Consider, in the pseudoultrametric space  $(V(P), \rho)$ , a cycle  $C$  with

$$V(C) = V(P), \quad E(C) = E(P) \cup \{\{v_1, v_2\}\}.$$

Then  $\{v_1, v_2\}$  is the unique edge of  $C$  on which  $\max\{\rho(x, y) : \{x, y\} \in E(C)\}$  is achieved, contrary to Lemma 2.1. Thus, for every  $v_1, v_2 \in V(G)$  and  $\rho \in \mathfrak{F}_{w,u}$  the inequality  $\rho(v_1, v_2) \leq \rho_w(v_1, v_2)$  holds, i.e.  $\rho_w$  is the greatest element of  $(\mathfrak{F}_{w,u}, \preceq)$ .  $\square$

**Remark 2.5.** If  $G$  is a finite graph and a weight  $w$  is defined by some metric as in (1.2), then the subdominant pseudoultrametric  $\rho_w$  is an ultrametric. For the complete  $G$  this classic case was considered in [5]. An efficient procedure for the evaluation of the subdominant ultrametric on the finite metric spaces can be found in [6] and [7].

We now turn to the study of connections between the shortest-path pseudometric and the subdominant pseudoultrametric. Remind that the shortest-path pseudometric is a pseudometric defined on  $V(G)$  as

$$d_w(x, y) = \begin{cases} 0, & \text{if } x = y \\ \inf_{P \in \mathfrak{P}_{x,y}} \sum_{e \in E(P)} w(e), & \text{if } x \neq y. \end{cases} \quad (2.8)$$

Similarly to Definition 2.3 we introduce

**Definition 2.6.** Let  $(G, w)$  be a nonempty weighted graph and  $\mathfrak{F}_{w,m}$  be the set of the pseudometrics  $d$  such that

$$d(u, v) \leq w(\{u, v\}) \quad (2.9)$$

for all edges  $\{u, v\} \in E(G)$ . The greatest element of the poset  $(\mathfrak{F}_{w,m}, \preceq)$  is called, if it exists, the subdominant pseudometric (for the weight  $w$ ).

**Proposition 2.7.** Let  $(G, w)$  be a nonempty connected weighted graph. Then  $d_w$  is the subdominant pseudometric for the weight  $w$ .

*Proof.* The inequality  $d_w(u, v) \leq w(\{u, v\})$  holds for every  $\{u, v\} \in E(G)$ . This follows from (2.8) and the fact that the two-term sequence  $u, v$  is a path belonging to  $\mathfrak{P}_{u,v}$ . Consequently,  $d_w \in \mathfrak{F}_{w,m}$ . Suppose that there are  $d \in \mathfrak{F}_{w,m}$  and  $u, p \in V(G)$  for which  $d(u, p) > d_w(u, p)$ . Then there exists a path  $(u = x_1, \dots, x_n = p) \in \mathfrak{P}_{u,p}$  such that

$$d(u, p) > \sum_{i=1}^{n-1} w(\{x_i, x_{i+1}\}). \quad (2.10)$$

Since  $d \in \mathfrak{F}_{w,m}$ , the inequality  $w(\{x_i, x_{i+1}\}) \geq d(x_i, x_{i+1})$  holds for  $i = 1, \dots, n-1$ . From here and (2.10) we find

$$d(x_1, x_n) \geq \sum_{i=1}^{n-1} d(x_i, x_{i+1}),$$

contrary to the triangle inequality.  $\square$

**Corollary 2.8.** *Let  $(G, w)$  be a nonempty connected weighted graph. Then the pseudoultrametric  $\rho_w$  constructed by rule (2.4) is the subdominant pseudoultrametric for  $d_w$ , i.e.,  $\rho_w \preceq d_w$  and  $\rho \preceq \rho_w$  for every pseudoultrametric  $\rho$  satisfying  $\rho \preceq d_w$ .*

*Proof.* Denote by  $\rho_w^*$  the subdominant pseudoultrametric for  $d_w$ . From (2.1), (2.8) and (2.4) it follows that  $\rho_w \preceq d_w$ , consequently  $\rho_w \preceq \rho_w^*$ . The inverse relation  $\rho_w^* \preceq \rho_w$  follows from the fact that every pseudoultrametric  $\rho$  satisfying  $\rho \preceq d_w$  belongs to the set  $\mathfrak{F}_{w,u}$  (see Definition 2.3). Thus,  $\rho_w^* = \rho_w$ .  $\square$

**Remark 2.9.** The question when  $d_w$  and  $\rho_w$  are metrics is of interest in its own right. We return to this in Section 4. Note that the problem of finding a criterion of existence of the subdominant **ultrametric**, for a given metric, was posed in [8].

### 3 Pseudoultrametrization of weighted graphs

**Definition 3.1.** *Let  $(G, w)$  be a weighted graph and let  $d : V(G) \times V(G) \rightarrow \mathbb{R}^+$  be an ultrametric (pseudoultrametric). We shall say that  $d$  extends  $w$  if (1.2) holds for every  $\{x, y\} \in E(G)$ .*

**Definition 3.2.** *The weight  $w$  is ultrametrizable (pseudoultrametrizable) if there exists an ultrametric (pseudoultrametric) extending  $w$ .*

The following theorem gives us a criterion of pseudoultrametrizability for a given weight  $w$ .

**Theorem 3.3.** *Let  $(G, w)$  be a nonempty weighted graph. The weight  $w$  is pseudoultrametrizable if and only if, for each cycle  $C \subseteq G$ , there exist at least two distinct edges  $e_1, e_2 \in E(C)$  such that*

$$w(e_1) = w(e_2) = \max_{e \in E(C)} w(e). \quad (3.1)$$

*If  $G$  is a connected graph and  $w$  is a pseudoultrametrizable weight, then the subdominant pseudoultrametric extends  $w$ .*

*Proof.* Lemma 2.1 implies that (3.1) holds for each cycle  $C \subseteq G$  if  $w$  is a pseudoultrametrizable weight.

Conversely, suppose that for each cycle  $C \subseteq G$  there exist  $e_1, e_2 \in E(C)$  such that (3.1) holds and prove the pseudoultrametrizability of  $w$ . Consider first the case, where  $G$  is connected. By Theorem 2.4 it is sufficient to prove, for every edge  $\{u, v\} \in E(G)$ , the following equality

$$\rho_w(u, v) = w(\{u, v\}), \quad (3.2)$$

where  $\rho_w$  is the subdominant pseudoultrametric for  $w$ . By Definition 2.3, we have

$$\rho_w(u, v) \leq w(\{u, v\}).$$

If this inequality is strict, then there exists a path  $P \in \mathfrak{P}_{u,v}$  for which

$$\max_{e \in P} w(e) < w(\{u, v\}).$$

The last inequality implies that  $\{u, v\} \notin E(P)$ . Since  $\{u, v\} \in E(G)$ , there exists a cycle  $C$  with

$$V(C) = V(P), \quad E(C) = E(P) \cup \{\{u, v\}\}.$$

If  $e_i$  is an edge of  $C$  different from  $\{u, v\}$ , then  $e_i \in E(P)$ , so that

$$w(e_i) \leq \max_{e \in P} w(e) < w(\{u, v\}) = \max_{e \in C} w(e),$$

contrary to (3.1). The pseudoultrametrizability of  $w$  is proved for the connected  $G$ .

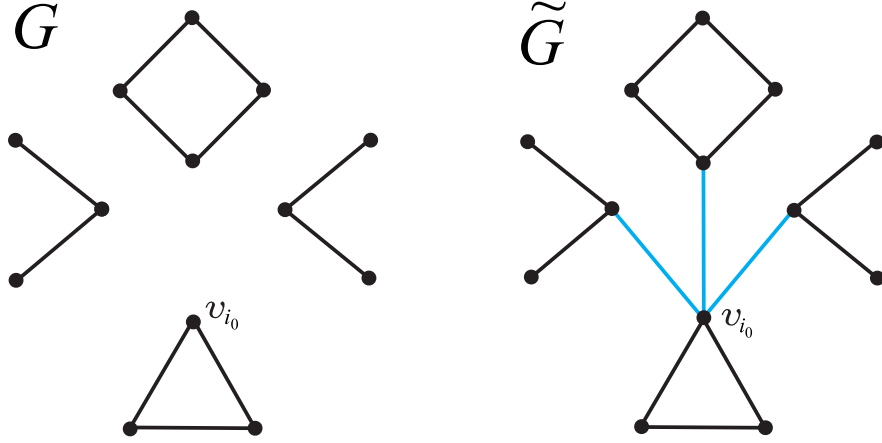


Figure 1: The transition from a disconnected graph  $G$  to a connected  $\tilde{G}$ .

Let  $G$  now be disconnected. Consider the set  $\{G_i : i \in \mathfrak{I}\}$  of connected components of  $G$ , where  $\mathfrak{I}$  is an indexing set. For each  $i \in \mathfrak{I}$  choose  $v_i \in V(G_i)$  and fix an index  $i_0 \in \mathfrak{I}$ . Consider a new graph  $\tilde{G}$  with

$$V(\tilde{G}) = V(G), \quad E(\tilde{G}) = E(G) \cup \{\{v_i, v_{i_0}\} : i \in \mathfrak{I} \setminus \{i_0\}\}.$$

Let us extend  $w$  to a function  $\tilde{w} : E(\tilde{G}) \rightarrow \mathbb{R}^+$  as

$$\tilde{w}(e) = \begin{cases} w(e) & \text{if } e \in E(G) \\ c_i & \text{if } e = \{v_i, v_{i_0}\}, i \in \mathfrak{I} \setminus \{i_0\} \end{cases} \quad (3.3)$$

where  $c_i$  are arbitrary non-negative constants. Since  $\tilde{G}$  is a connected graph, to prove the pseudoultrametrizability of  $G$  it is sufficient to establish (3.1) with  $w = \tilde{w}$  for every cycle  $C \subseteq \tilde{G}$ . Observe that each cycle  $C \subseteq \tilde{G}$  is a cycle in  $G$ . It is easily seen by drawing an appropriate picture (see Fig. 1). The formal proof is as follows. Let  $C \subseteq \tilde{G}$  but  $C \not\subseteq G$ . Then there exists an edge in  $C$  of the form  $\{v_{i_0}, v_{i_1}\}$ ,  $i_1 \in \mathfrak{I} \setminus \{i_0\}$  and there exists a unique edge incident to  $v_{i_0}$  in  $C$  and different from  $\{v_{i_0}, v_{i_1}\}$ . The definition

of  $\tilde{G}$  implies that this edge has the form  $\{v_{i_0}, v_{i_2}\}$ . Removing, from the cycle  $C$ , the vertex  $v_{i_0}$  we get the path  $P$ ,

$$V(P) = V(C) \setminus \{v_{i_0}\}, \quad E(P) = E(C) \setminus \{\{v_{i_0}, v_{i_1}\}, \{v_{i_0}, v_{i_2}\}\},$$

connecting  $v_{i_1}$  and  $v_{i_2}$ . Since  $E(P) \subseteq E(G)$ ,  $v_{i_1}$  and  $v_{i_2}$  lie in the same connected component, which contradicts to their definition. Consequently if  $C \subseteq \tilde{G}$ , then  $C \subseteq G$ , so that (3.1) follows.  $\square$

**Remark 3.4.** Condition (3.1) is equivalent to the strong triangle inequality if the cycle  $C$  contains exactly three vertices. Note also that

$$\sum_{e \in E(C)} w(e) \geq w(e_1) + w(e_2) = 2 \max_{e \in E(C)} w(e)$$

if (3.1) holds. Thus, the condition of the pseudoultrametrizability of the weight implies the condition of its pseudometrization, as expected.

**Remark 3.5.** Directly from Theorem 3.3 it follows that the pseudoultrametrizability of the weight is a local property, i.e., if for every **finite** subgraph  $H$  of a weighted graph  $(G, w)$  the restriction of  $w$  on  $E(H)$  is pseudoultrametrizable, then the weight  $w$  is pseudoultrametrizable too.

Recall that a graph which does not contain any cycles is called a *forest* and a *tree* is a connected forest.

**Corollary 3.6.** *Let  $G$  be a graph with  $V(G) \neq \emptyset$ .  $G$  is a forest if and only if every weight  $w : E(G) \rightarrow \mathbb{R}^+$  is pseudoultrametrizable.*

For the proof it suffices to note that the existence of a cycle  $C \subseteq G$  implies the existence of a weight  $w : E(G) \rightarrow \mathbb{R}^+$  for which condition (3.1) does not hold.

Let  $(G, w)$  be a weighted graph with a pseudoultrametrizable weight  $w$ . Denote by  $\mathfrak{U}_w$  the set of all pseudoultrametrics on  $V(G)$  extending  $w$ .

**Theorem 3.7.** *If  $G$  is connected, then the subdominant pseudoultrametric  $\rho_w$  is the greatest element of the poset  $(\mathfrak{U}_w, \preceq)$ . Conversely, if  $(\mathfrak{U}_w, \preceq)$  has the greatest element, then  $G$  is connected.*

*Proof.* Let  $\mathfrak{F}_{w,u}$  be the set from Definition 2.3. Suppose  $G$  is a connected graph. Then  $\mathfrak{F}_{w,u} \supseteq \mathfrak{U}_w$  and the subdominant pseudoultrametric  $\rho_w$  belongs to  $\mathfrak{F}_{w,u}$ . By the definition of the subdominant pseudoultrametric we have  $\rho \preceq \rho_w$  for every  $\rho \in \mathfrak{U}_w$ . To prove that  $\rho_w$  is the greatest element of  $(\mathfrak{U}_w, \preceq)$  it is sufficient to verify the relation  $\rho_w \in \mathfrak{U}_w$  that has already been established in Theorem 3.3.

Suppose  $G$  is not connected. Fix some points  $v_{i_0}$  and  $v_{i_1}$  belonging to distinct connected components. Let us consider the weighted graph  $(\tilde{G}, \tilde{w})$  as it was done in the



proof of Theorem 3.3. It is clear that  $\mathfrak{U}_w \supseteq \mathfrak{U}_{\tilde{w}}$ . The last inclusion and the arbitrariness of constants  $c_i$  in (3.3) imply the equality

$$\sup_{\rho \in \mathfrak{U}_w} \rho(v_{i_0}, v_{i_1}) = +\infty.$$

Consequently, the poset  $(\mathfrak{U}_w, \preceq)$  does not contain the greatest element for the disconnected graphs.  $\square$

Using the last theorem we can easily obtain the converse assertion to Corollary 2.8.

**Corollary 3.8.** *Let  $(G, w)$  be a nonempty weighted graph. If, for  $w$ , there exists the subdominant pseudoultrametric, then  $G$  is connected.*

**Remark 3.9.** In corollaries 2.8 and 3.8 we do not require the pseudoultrametrizability of  $w$ .

In order to trace an analogy between  $\rho_w$  and  $d_w$ , denote by  $\mathfrak{M}_w$  the family of the pseudometrics extending  $w$ .

As it was shown in [2] for connected  $G$ , the shortest-path pseudometric  $d_w$  belongs to  $\mathfrak{M}_w$  for every pseudometrizable weight  $w$ . If we introduce a partial order  $\preceq$  in  $\mathfrak{M}_w$  as on the subset of  $(\mathfrak{F}, \preceq)$  (see (2.3)), then the following analog of Theorem 3.7 holds.

**Theorem 3.10** ([2]). *Let  $(G, w)$  be a nonempty weighted graph with the pseudometrizable weight  $w$ . If  $G$  is connected, then  $d_w$  is the greatest element of the poset  $(\mathfrak{M}_w, \preceq)$ . Conversely, if  $(\mathfrak{M}_w, \preceq)$  has a greatest element, then the graph  $G$  is connected.*

Theorems 3.7 and 3.10 imply

**Corollary 3.11.** *Let  $G$  be a nonempty graph. The following statements are equivalent:*

- (i)  $G$  is a connected graph;
- (ii) The poset  $(\mathfrak{M}_w, \preceq)$  has the greatest element for every pseudometrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$ ;
- (iii) The poset  $(\mathfrak{U}_w, \preceq)$  has the greatest element for every pseudoultrametrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$ .

Using Corollary 3.6, Theorem 3.7, and the corresponding results from [2] we get

**Corollary 3.12.** *Let  $G$  be a nonempty graph. The following statements are equivalent:*

- (i)  $G$  is a tree;
- (ii) Every weight  $w : E(G) \rightarrow \mathbb{R}^+$  is pseudometrizable and the poset  $(\mathfrak{M}_w, \preceq)$  contains the greatest element;

(iii) Every weight  $w : E(G) \rightarrow \mathbb{R}^+$  is pseudoultrametrizable and the poset  $(\mathfrak{U}_w, \preceq)$  contains the greatest element.

Another examples illustrating the analogy between  $\rho_w$  and  $d_w$  are given in the next section.

**Remark 3.13.** If the weight  $w$  is pseudometrizable but not pseudoultrametrizable, then the following problem arises. Find the extension of  $w$  being as “ultrametrizable” as possible.

We can introduce an appropriate measure of “ultrametrizability” using the so-called “betweenness exponent” being the supremum  $\alpha \geq 1$  for which  $d^\alpha$  remains to be a metric for a given metric  $d$  (see [9], [10]).

We can easily extend the notion of betweenness exponent to the case of weighted graphs. If  $\alpha = 1$ , then the betweenness exponent gives us the condition of pseudometrizable of the weight  $w$  and, if  $\alpha = \infty$ , the condition of pseudoultrametrizability of  $w$ .

## 4 Ultrametrization of weighted graphs

In the previous section it was proved that a weight  $w : E(G) \rightarrow \mathbb{R}^+$  is pseudoultrametrizable if and only if condition (3.1) holds for every cycle  $C \subseteq G$ . If a pseudoultrametrizable weight  $w$  is strictly positive, i.e., for every  $e \in E(G)$  we have

$$w(e) > 0,$$

and  $G$  is finite and connected, then it is clear that the subdominant pseudoultrametric  $\rho_w$  is an ultrametric. The following example shows that, for infinite  $G$ , the strict positivity of  $w$  does not guarantee that  $\rho_w$  is an ultrametric.

**Example 4.1.** Let  $(G, w)$  be an infinite weighted graph, depicted in Figure 2, where

$$\varepsilon_n = w(\{u, s_n\}) = w(\{s_n, t_n\}) = w(\{t_n, v\})$$

are positive real numbers such that  $\lim_{n \rightarrow \infty} \varepsilon_n = 0$  and  $\varepsilon_n > \varepsilon_{n+1}$  for each  $n$ . The length of every cycle  $C \subseteq G$  is equal to six and its vertices are  $u, s_n, t_n, v, t_m, s_m$ , where  $m \neq n$ . Such cycle  $C$  has the three distinct edges of maximal weight. Consequently,  $w$  is pseudoultrametrizable. The definition of  $\rho_w$  implies that

$$\rho_w(u, v) = \max\{w(\{u, s_n\}), w(\{s_n, t_n\}), w(\{t_n, v\})\} = \varepsilon_n \vee \varepsilon_m.$$

Letting  $n, m \rightarrow \infty$ , we obtain  $\rho_w(u, v) = 0$ .

From Theorem 3.7 it follows, for connected  $G$ , that the set  $\mathfrak{U}_w$  contains ultrametries if and only if  $\rho_w$  is an ultrametric. In the present section we shall describe the structure of the graphs  $G$  for which  $\rho_w$  is an ultrametric for every strictly positive pseudoultrametrizable weight  $w$ .

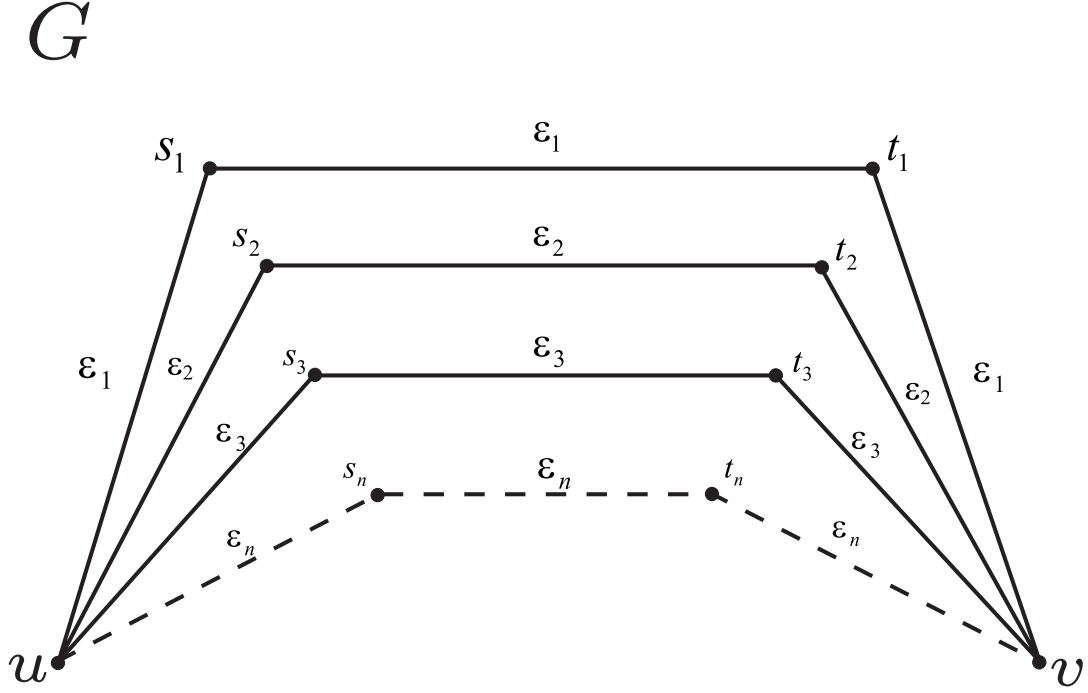


Figure 2: The weighted graph with the pseudoultrametrizable weight  $w$  such that  $\mathfrak{U}_w$  does not contain ultrametrics.

Note that the paper [11] contains the complete characterization of metric spaces  $(X, d)$  for which the subdominant (for  $d$ ) pseudoultrametric is an ultrametric.

We need the following lemma from [2].

**Lemma 4.2.** *Let  $G$  be a connected graph and let  $u^*, v^*$  be two nonadjacent vertices of  $G$ . Let  $\tilde{F} = \{F_j\}_{j \in \mathbb{N}}$  be a sequence of paths connecting  $u^*$  and  $v^*$  and meeting the following condition:*

(i<sub>1</sub>) *For every  $e^0 \in E(G)$  there exist  $u^0 \in e^0$  and  $i = i(e^0)$  such that  $u^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k})$ .*

*Then there exists a subsequence  $\{F_{j_k}\}_{k \in \mathbb{N}}$  of the sequence  $\tilde{F}$  such that:*

(i<sub>2</sub>)  $E(F_{j_l}) \cap E(F_{j_k}) = \emptyset$  for  $l \neq k$ ;

(i<sub>3</sub>) *If  $C$  is a cycle in the graph  $\bigcup_{k \in \mathbb{N}} F_{j_k}$  and*

$$k_0 = k_0(C) := \min\{k \in \mathbb{N} : E(C) \cap E(F_{j_k}) \neq \emptyset\}, \quad (4.1)$$

*then  $C$  and  $F_{j_{k_0}}$  have at least two common edges.*

**Remark 4.3.** Here and below by the union  $\bigcup_{i \in \mathcal{I}} G_i$  of subgraphs  $G_i$  of the graph  $G$  we shall always mean the subgraph  $\tilde{G} \subseteq G$  for which

$$V(\tilde{G}) = \bigcup_{i \in \mathcal{I}} V(G_i) \text{ and } E(\tilde{G}) = \bigcup_{i \in \mathcal{I}} E(G_i).$$

The next theorem is the main result of the section.

**Theorem 4.4.** *Let  $G = (V, E)$  be a nonempty connected graph. The following two statements are equivalent.*

- (i) *For every strictly positive pseudoultrametrizable weight  $w$  the subdominant pseudoultrametric  $\rho_w$  is an ultrametric.*
- (ii) *For every pair of distinct points  $u^*, v^* \in V(G)$  and for an arbitrary sequence  $\tilde{F}$  of paths  $F_j \in \mathfrak{P}_{u^*, v^*}$ ,  $j \in \mathbb{N}$  there exists an edge  $e^0 = \{u^0, v^0\} \in E(G)$  such that*

$$u^0, v^0 \in \bigcup_{k=1}^{\infty} V(F_{i+k}) \text{ for every } i > 0.$$

*Proof.* (i)  $\Rightarrow$  (ii) Suppose that statement (ii) does not hold. Then there exist a pair of distinct vertices  $u^*, v^*$  and a sequence  $\tilde{F}$  of paths  $F_j \in \mathfrak{P}_{u^*, v^*}$  such that for every edge  $e^0 \in E(G)$ ,  $e^0 = \{u^0, v^0\}$ , there exist  $i \in \mathbb{N}$  and at least one vertex incident to  $e^0$ , for example  $u^0$ , for which

$$u^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k}).$$

Let us show that, in this case, statement (i) does not hold. By Lemma 4.2, without loss of generality, it can be assumed that

$$E(F_i) \cap E(F_j) = \emptyset \text{ for } i \neq j, \quad (4.2)$$

and every cycle  $C$  from  $\bigcup_{j \in \mathbb{N}} F_j$  has at least two common edges with  $F_{k_0}$ , where  $k_0$  is defined as in (4.1). Consider the graph

$$\tilde{G} = \bigcup_{i \in \mathbb{N}} F_i.$$

Let  $\{\varepsilon_i\}_{i \in \mathbb{N}}$  be a strictly decreasing sequence of positive real numbers with  $\lim_{i \rightarrow \infty} \varepsilon_i = 0$ . Define the weight  $w_1 : E(\tilde{G}) \rightarrow \mathbb{R}^+$  as  $w_1(e) := \varepsilon_i$  if  $e \in E(F_i)$ . The definition is correct by virtue of (4.2). All edges of the path  $F_i$ ,  $i \in \mathbb{N}$  have the same weight  $\varepsilon_i$  and every edge of the path  $F_{i+1}$  has a weight strictly less than  $\varepsilon_i$ . Every cycle  $C \subseteq \tilde{G}$  has at least two common edges  $e_1, e_2$  with the path  $F_{k_0}$ , where  $k_0$  is defined by (4.1). Consequently the weights of these edges are maximal,  $w_1(e_1) = w_1(e_2) = \varepsilon_{k_0} = \max_{e \in E(C)} w_1(e)$ . By Theorem 3.3, the weight  $w_1$  is pseudoultrametrizable. Choosing the pseudoultrametrization  $\rho_{w_1}$  as in (2.4) we get

$$\rho_{w_1}(u^*, v^*) = \inf_{i \in \mathbb{N}} (\max_{e \in F_i} w(e)) = \inf_{i \in \mathbb{N}} \varepsilon_i = 0. \quad (4.3)$$

Thus  $\rho_{w_1}$  is not metric on the set  $V(\tilde{G})$ .

Using the pseudoultrametric  $\rho_{w_1}$  we extend the weighted function  $w_1$  to the set of edges of  $G$  having the vertices in  $V(\tilde{G})$ . Let us prove that we obtain again a strictly positive weight (for which we keep the same notation  $w_1$ ). Let  $e_0 = \{u_0, v_0\} \in E(G)$ ,  $u_0 \in V(\tilde{G})$  and  $v_0 \in V(\tilde{G})$ . By assumption there is at least one end of the edge  $e^0$ , for example  $u^0$ , and there exists an index  $i_0$  such that

$$u^0 \notin V(F_i) \quad (4.4)$$

if  $i > i_0$ . Let  $F$  be a path in  $\tilde{G}$  connecting  $u^0$  and  $v^0$  and let  $e \in E(F)$  be an edge incident with  $u^0$ . From (4.4) it follows that

$$e \in \bigcup_{i=1}^{i_0} E(F_i).$$

Since the sequence  $\{\varepsilon_i\}_{i \in \mathbb{N}}$  is decreasing, we get  $w(e) \geq \varepsilon_{i_0}$ , so that  $\max_{e \in F} w(e) \geq \varepsilon_{i_0} > 0$ . Thus

$$w_1(\{u_0, v_0\}) = \rho_{w_1}(u^0, v^0) > 0.$$

The next step of our proof is to assign some positive weights for edges  $e = \{u, v\} \in E(G)$  having  $u \notin V(\tilde{G})$  or  $v \notin V(\tilde{G})$ . Assign for every such edge  $e = \{u, v\}$  the weight  $w_2(e) = M$ , where  $M$  is an arbitrary number from  $[\varepsilon_1, \infty)$ . Moreover if  $e = \{u, v\}$  with  $u, v \in V(\tilde{G})$ , then set  $w_2(e) := \rho_{w_1}(u, v)$ . It is clear that so defined weight  $w_2$  is pseudoultrametrizable and  $w_2(e) > 0$  for every  $e \in E(G)$ . Indeed, if  $C$  is a cycle in  $G$  having all the edges in  $\tilde{G}$ , then there are two maximum weight edges in  $C$  because  $\rho_{w_1}$  is a pseudoultrametric. Now let  $v \in V(C)$  but  $v \notin V(\tilde{G})$ . Then two edges of the cycle  $C$  which are incident with  $v$  have the maximal weight  $M$ . The pseudoultrametrizability of  $w_2$  follows from Theorem 3.3. Since  $\tilde{G} \subseteq G$  and  $w_1 = w_2|_{E(\tilde{G})}$ , from (2.4) we obtain

$$\rho_{w_2}(u, v) \leq \rho_{w_1}(u, v) \quad (4.5)$$

for  $u, v \in V(\tilde{G})$ . Relations (4.3) and (4.5) imply  $\rho_{w_2}(u^*, v^*) = 0$ . Thus we have found the strictly positive pseudoultrametrizable weight  $w_2$  for which the subdominant pseudoultrametric  $\rho_{w_2}$  is not an ultrametric.

(ii)  $\Rightarrow$  (i) Let  $(G, w)$  be a weighted graph with a pseudoultrametrizable strictly positive weight and let condition (ii) hold. Let us prove that the pseudoultrametric  $\rho_w : V(G) \times V(G) \rightarrow \mathbb{R}^+$  is an ultrametric.

Suppose the contrary. Then, for some vertices  $u^*$  and  $v^*$ ,  $u^* \neq v^*$ , we have  $\rho_w(u^*, v^*) = 0$ . Consequently there exists a sequence  $\{F_k\}_{k \in \mathbb{N}}$ ,  $F_k \in \mathfrak{P}_{u^*, v^*}$ , such that for every  $\varepsilon > 0$  there exists  $k(\varepsilon) \in \mathbb{N}$  meeting the inequality

$$\max_{e \in F_k} w(e) < \varepsilon \quad (4.6)$$

for  $k \geq k(\varepsilon)$ . By assumption (ii) there exists an edge  $e^0 = \{u^0, v^0\} \in E(G)$ , such that  $u^0, v^0 \in \bigcup_{i=1}^{\infty} V(F_{i+k})$  for all  $k > 0$ .

Let us choose a path  $P$  connecting  $u^0$  and  $v^0$  in the graph  $G_\varepsilon := \bigcup_{i=1}^{\infty} F_{k(\varepsilon)+i}$ . The definition of  $G_\varepsilon$  and (4.6) imply the inequality

$$w(e) < \varepsilon$$

for every  $e \in P$ . This inequality and the definition of  $\rho_w$  give us

$$\rho_w(u^0, v^0) \leq \max_{e \in P} w(e) < \varepsilon.$$

Letting  $\varepsilon$  to zero, we obtain  $\rho_w(u^0, v^0) = 0$ . Since  $w$  is strictly positive and pseudoultrametrizable, by Theorem 3.3 we have

$$0 < w(\{u^0, v^0\}) = \rho_w(u^0, v^0).$$

This contradiction completes the proof.  $\square$

This theorem and the corresponding result from [2] imply

**Corollary 4.5.** *Let  $G$  be a nonempty connected graph. The following two statements are equivalent:*

- (i) *The subdominant pseudoultrametric  $\rho_w$  is an ultrametric for every strictly positive pseudoultrametrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$ ;*
- (ii) *The shortest-path pseudometric  $d_w$  is a metric for every strictly positive pseudometrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$ .*

Using theorems 3.7 and 4.4 it is simple to describe the structural properties of graphs  $G$  for which there exist strictly positive pseudoultrametrizable weights  $w$  such that  $\mathfrak{U}_w$  does not contain any ultrametric. Recall some definitions.

For a sequence of the sets  $A_n$ ,  $n \in \mathbb{N}$ , the *upper limit*,  $\limsup_{n \rightarrow \infty} A_n$ , is the set of elements  $a$  such that  $a \in A_n$  for infinitely many  $n$ , i. e.,

$$\limsup_{n \rightarrow \infty} A_n = \bigcap_{k=1}^{\infty} \left( \bigcup_{n=1}^{\infty} A_{n+k} \right).$$

The subset  $V_0$  of the vertices set of a graph  $G$  is called *independent*, if every two vertices from  $V_0$  are not adjacent.

**Corollary 4.6.** *Let  $G = (V, E)$  be a nonempty connected graph. The following two statements are equivalent:*

- (i) *There exists a strictly positive pseudoultrametrizable weight  $w$  such that the set  $\mathfrak{U}_w$  does not contain any ultrametric;*

(ii) *There exist some vertices  $u^*, v^* \in V(G)$  and a sequence  $\{F_j\}_{j \in \mathbb{N}}$ ,  $F_j \in \mathfrak{P}_{u^*, v^*}$ , such that*

$$\limsup_{j \rightarrow \infty} V(F_j)$$

*is an independent set.*

Now we shall give some examples of graphs  $G$  for which every strictly positive pseudoultrametrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$  can be extended to an ultrametric.

**Example 4.7.** If every connected component of nonempty graph  $G = (V, E)$  contains at most one vertex of the infinite degree, then for each strictly positive pseudoultrametrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$  there exists an ultrametric  $\rho \in \mathfrak{U}_w$ .

Indeed, let  $\{G_i : i \in \mathcal{I}\}$  be the set of connected components of  $G$ . We complete the graph  $G$ , if necessary, to the connected graph  $\tilde{G}$  as it was done in the proof of Theorem 3.3. If distinct vertices  $u^*$  and  $v^*$  lie in the same component  $G_i$ , then one of them, for example  $u^*$ , is incident to the finite number of edges  $e_1, e_2, \dots, e_n$ . Thus, for an arbitrary infinite sequence of paths  $F_i \in \mathfrak{P}_{u^*, v^*}$ , one from  $e_j$ ,  $j = 1, \dots, n$  belongs to the infinite number of paths.

If an arbitrary vertices  $u^*$  and  $v^*$  lie in the different connected components, then there exists an edge of the form  $\{v_{i_0}, v_i\}$  which belongs to every path  $F_i \in \mathfrak{P}_{u^*, v^*}$ . In both cases, applying Theorem 4.4, we get the existence of an ultrametric  $\rho \in \mathfrak{U}_w$ .

**Example 4.8.** If the nonempty graph  $G$  is a tree, then the subdominant pseudoultrametric  $\rho_w$  is an ultrametric for each strictly positive weight  $w : E(G) \rightarrow \mathbb{R}^+$ . Indeed, the well-known characteristic property of trees says that for every two distinct vertices  $u^*, v^* \in V(G)$  there exists only one path connected them in  $G$ . Consequently, every sequence  $\tilde{F}$  of paths  $F_j \in \mathfrak{P}_{u^*, v^*}$  is stationary,  $F_1 = F_2 = \dots = F_n = F_{n+1} = \dots$ . This leads to the automatic truth of assertion (ii) from Theorem 4.4.

## 5 The least element of $\mathfrak{U}_w$ and uniqueness of pseudoultrametric extension of weight

In the present section we shall show that only the complete  $k$ -partite graphs have the following property: the poset  $(\mathfrak{U}_w, \preceq)$  contains the least element for every pseudoultrametrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$ . Moreover, a uniqueness criterion for the problem of the extension of a weight  $w$  to a pseudoultrametric  $\rho : V(G) \times V(G) \rightarrow \mathbb{R}^+$  will be given also. The criterion of existence of such extensions was obtained above in Theorem 3.3.

Recall that a graph  $G$  is called  $k$ -partite, if the set  $V(G)$  can be decomposed into  $k$  ( $k$  is an arbitrary cardinal number) nonempty disjoint sets  $V_\alpha$ ,

$$V(G) = \bigcup_{\alpha \in I} V_\alpha, \quad \alpha \in I, \quad |I| = k, \quad V_{\alpha_i} \cap V_{\alpha_j} = \emptyset \text{ if } i \neq j,$$

such that for every  $\{x, y\} \in E(G)$  the vertices  $x$  and  $y$  lie in distinct parts  $V_\alpha$ . A  $k$ -partite graph is *complete* if every two vertices in distinct parts are adjacent. It is clear that a  $k$ -partite graph is empty, if  $k = 1$  and connected, if  $k \geq 2$ .



Figure 3: If we define a weight  $w$  on  $E(H)$  such that  $w(\{u, v\}) > 0$ , then  $(\mathfrak{U}_w, \preceq)$  does not contain the least element.

The proof of the following lemma can be found in [12].

**Lemma 5.1.** *Let  $G$  be a graph with  $V(G) \neq \emptyset$ . Then  $G$  is a complete  $k$ -partite with  $k \geq 1$  if and only if  $G$  does not contain any induced subgraphs isomorphic to the graph  $H$  depicted by Figure 3.*

We shall denote by TM (twice-max) the set of unordered pairs  $p, q$  of distinct nonadjacent vertices of the graph  $(G, w)$  having the following property: **each** path  $P \in \mathfrak{P}_{p,q}$  contains at least two distinct edges  $e_1$  and  $e_2$  such that  $w(e_1) = w(e_2) = \max_{e \in E(P)} w(e)$ .

**Theorem 5.2.** *The following conditions are equivalent for every nonempty graph  $G$ :*

- (i) *The poset  $(\mathfrak{U}_w, \preceq)$  contains the least pseudoultrametric  $\rho_{0,w}$  for every pseudoultrametrizable weight  $w : E(G) \rightarrow \mathbb{R}^+$ , i.e., the inequality*

$$\rho_{0,w}(u, v) \leq \rho(u, v) \tag{5.1}$$

*holds for every  $\rho \in \mathfrak{U}_w$  and all  $u, v \in V(G)$ ;*

- (ii)  *$G$  is a complete  $k$ -partite graph with  $k \geq 2$ .*

*If condition (ii) holds and  $w$  is a pseudoultrametrizable weight, then for  $u \neq v$  we have*

$$\rho_{0,w}(u, v) = \begin{cases} 0 & \text{if } \{u, v\} \in \text{TM} \\ \max_{e \in E(F)} w(e) & \text{if } \{u, v\} \notin \text{TM} \end{cases} \tag{5.2}$$

*where  $F$  is an arbitrary path from  $\mathfrak{P}_{u,v}$  for which  $\max_{e \in E(F)} w(e)$  is achieved on a single edge.*



*Proof.* **(i)  $\Rightarrow$  (ii)** Suppose (ii) does not hold. Then, by Lemma 5.1, there exist vertices  $u, v$ ,  $\{u, v\} \in E(G)$ , and a vertex  $p \in V(G)$ ,  $u \neq p \neq v$  such that  $\{u, p\} \notin E(G)$  and  $\{v, p\} \notin E(G)$ . Define the weight  $w(e) = 1$  for every  $e \in E(G)$ . Consider the following two pseudoultrametrics on the set  $V(G)$ :

$\rho_1(u, p) = \rho_1(p, u) = \rho_1(s, s) = 0$  for all  $s \in V(G)$  and  $\rho_1(s, t) = 1$  in the opposite case;

$\rho_2(v, p) = \rho_2(p, v) = \rho_2(s, s) = 0$  for all  $s \in V(G)$  and  $\rho_2(s, t) = 1$  in the opposite case.

It is clear that  $\rho_1, \rho_2 \in \mathfrak{U}_w$ . Assuming that there exists the least pseudoultrametric  $\rho \in \mathfrak{U}_w$ , we get the contradiction

$$1 = \rho(u, v) \leq \rho(u, p) + \rho(p, v) \leq (\rho_1 \wedge \rho_2)(u, p) + (\rho_1 \wedge \rho_2)(p, v) = 0.$$

The implication (i) $\Rightarrow$ (ii) follows.

**(ii)  $\Rightarrow$  (i)** Let condition (ii) hold and  $(G, w)$  be an arbitrary weighted graph with a pseudoultrametrizable  $w$ . Since  $k \geq 2$ ,  $G$  is a connected graph, as it was mentioned above. Let us show that  $\rho_{0,w}$  defined by (5.2) is the least element of the poset  $\mathfrak{U}_w$ .

Note that the function  $\rho_{0,w}$  is well defined. Indeed, suppose that there exist  $\{u, v\} \notin \text{TM}$  and two distinct paths  $F_1, F_2 \in \mathfrak{P}_{u,v}$  such that each of them contains only one edge of maximal weight. Let  $\rho \in \mathfrak{U}_w$ . Consider in the pseudoultrametric space  $(V(G), \rho)$  the cycle generated by the path  $F_1$  and the edge  $\{u, v\}$ . By Lemma 2.1 we obtain that  $\rho(u, v) = \max_{e \in E(F_1)} w(e)$ . Similarly if we consider the cycle generated by  $F_2$  and  $\{u, v\}$ , then  $\rho(u, v) = \max_{e \in E(F_2)} w(e)$ .

At the same time for every edge  $\{u, v\} \in E(G)$  we have  $\rho_{0,w}(u, v) = w(\{u, v\})$ , since in this case  $\{u, v\} \notin \text{TM}$  and the path  $(u, v)$  is one from the paths connecting the vertices  $u$  and  $v$ .

Let us prove that the function  $\rho_{0,w}$  is actually a pseudoultrametric. It is sufficient to establish the strong triangle inequality

$$\rho_{0,w}(x, y) \leq \rho_{0,w}(x, z) \vee \rho_{0,w}(z, y) \quad (5.3)$$

for pairwise distinct vertices  $x, y, z \in V(G)$ .

If all three points  $x, y, z$  are pairwise adjacent, then (5.3) follows from the pseudoultrametrizability of  $w$ . Let us show that (5.3) holds if among the vertices there are only two adjacent pairs. If the weights of corresponding edges are distinct, then, according to (5.2), the weight of the missing edge will be equal to the maximum weight of those edges. The inequality (5.3) holds again. Assume that only two pairs of vertices are adjacent and  $w(\{x, z\}) = w(\{z, y\}) = \rho_{0,w}(x, z) = \rho_{0,w}(z, y) = a$  but  $\rho_{0,w}(x, y) = b > a$ . Then there exists a path  $F \in \mathfrak{P}_{x,y}$  with a unique edge  $e_0$  having the maximal weight. Consider the cycle consisting of the path  $F$  and the edges  $\{x, z\}$  and  $\{z, y\}$  (or one of these edges, depending on whether the path  $F$  contains one of them). This cycle

contains the unique edge  $e_0$  of maximal weight, contrary to pseudoultrametrizability of  $w$  (see Theorem 3.3).

By Lemma 5.1 the case when exactly two vertices are adjacent is impossible. Therefore we may suppose that the vertices  $x, y, z$  are pairwise non-adjacent. Let  $\rho_{0,w}(x, y) = a > 0$  (if  $\rho_{0,w}(x, y) = 0$ , then inequality (5.3) is evident). Since  $a > 0$ , there exists a path  $F \in \mathfrak{P}_{x,y}$  with the unique edge  $e_0 = \{u_0, v_0\}$  of maximal weight  $w(e_0) = a$ . Suppose first that the path  $F$  does not pass through the point  $z$ . From (ii) it follows that at least one from the points  $u_0, v_0$ , for example  $u_0$ , is adjacent with  $z$ . Consider the following two paths:  $F_1 \in \mathfrak{P}_{x,z}$  consisting of the edge  $\{u_0, z\}$  and some part of the path  $F$  and the path  $F_2 \in \mathfrak{P}_{y,z}$  also consisting of the edge  $\{u_0, z\}$  and the rest of  $F$ . We may assume, without loss of generality, that  $\{u_0, v_0\} \in E(F_2)$ . If  $w(\{u_0, z\}) > a$ , then  $\{u_0, z\}$  is the unique edge of maximal weight for both  $F_1, F_2$ . Hence, from (5.2) we have

$$\rho_{0,w}(x, z) = w(\{u_0, z\}) = \rho_{0,w}(z, y) > a,$$

so that, (5.3) follows. Let  $w(\{u_0, z\}) \leq a$ . If the last inequality is strict, then  $\{u_0, v_0\}$  is the unique edge of maximal weight in  $F_2$ . Consequently

$$\rho_{0,w}(y, z) = w(\{u_0, v_0\}) = a,$$

so that (5.3) holds. If  $w(\{u_0, z\}) = a$ , then  $\{u_0, z\}$  is the unique edge of maximal weight in  $F_1$  and again we obtain (5.3). Consider the case when the path  $F$  passes through the point  $z$ . By splitting  $F$  into the two paths  $F_1 \in \mathfrak{P}_{x,z}$  and  $F_2 \in \mathfrak{P}_{z,y}$ , we obtain that one of the values  $\rho_{0,w}(x, z)$ ,  $\rho_{0,w}(z, y)$  is equal to  $a$ . Thus, (5.3) follows again.

It remains to prove that inequality (5.1) holds for every  $\rho \in \mathfrak{U}_w$  and  $u, v \in V(G)$ . This is trivial, if  $\rho_{0,w}(u, v) = 0$ . Assume that  $\rho_{0,w}(u, v) = a > 0$ , then  $\{u, v\} \notin \text{TM}$  and there exists a path  $F \in \mathfrak{P}_{u,v}$  with the unique edge  $e_0$  of maximal weight  $w(e_0) = a$ . Suppose that  $\rho(u, v) < a$ . Then the function  $\rho(x, y)$  is not a pseudoultrametric because the cycle  $C$  with  $V(C) = V(P)$  and  $E(C) = E(P) \cup \{\{u, v\}\}$  contains exactly one edge of maximal weight.  $\square$

As an application of Theorem 5.2 we shall obtain a characterization of the stars. Recall that a *star* is a complete bipartite graph for which at least one of the parts is a singleton.

**Corollary 5.3.** *The following conditions are equivalent:*

- (i) *Every weight  $w : E(G) \rightarrow \mathbb{R}^+$  is pseudoultrametrizable and the poset  $(\mathfrak{U}_w, \preceq)$  contains the least element;*
- (ii)  *$G$  is a star.*

*Proof.* The implication (ii)  $\Rightarrow$  (i) follows from Theorem 5.2 and Corollary 3.12. Let (i) hold, then again using Theorem 5.2 and Corollary 3.12 we obtain that  $G$  is a complete  $k$ -partite graph with  $k \geq 2$  being at the same time a tree. If  $k \geq 3$ , then  $G$  contains a triangle. To construct it, we may take three points lying in three distinct parts of the graph  $G$ . Since the trees do not contain cycles,  $k = 2$ . If each part of  $G$  contains at least two points, then it is easy to construct a quadruple (4-cycle)  $C \subseteq G$ , contrary again to the acyclic property of  $G$ . Thus,  $G$  is a complete bipartite graph one part of which is a singleton.  $\square$

We turn now to the conditions of uniqueness in the problem of extension of a weight  $w$  by pseudoultrametrics. We need the following

**Definition 5.4.** Let  $(G, w)$  be a nonempty weighted graph and let  $u, v$  be two distinct disjoint vertices of  $G$ . We shall say that  $u$  and  $v$  are well chained if for every  $\varepsilon > 0$  there exists a path  $u = u_1, u_2, \dots, u_n = v$  such that  $\{u_i, u_{i+1}\} \in E(G)$  and  $w(\{u_i, u_{i+1}\}) \leq \varepsilon$  for  $i = 1, \dots, n - 1$ .

We denote the set of all such pairs  $\{u, v\}$  by WCh.

**Remark 5.5.** The notion “well chained points” is often used in the metric continuum theory [13, c. 60] and it plays an important role in the considering of problems related to the connectivity in metric spaces (see, e.g., [14]).

**Remark 5.6.** The notion “well chained points” arises naturally in the study of subdominant ultrametrics (see [7] and [11]). In particular, it is easy to show that, for a strictly positive pseudoultrametrizable weight  $w$  and a connected graph  $G$ , some vertices  $u$  and  $v$ ,  $u \neq v$  are well chained if and only if  $\rho_w(u, v) = 0$ .

**Theorem 5.7.** Let  $(G, w)$  be a nonempty connected weighted graph with a pseudoultrametrizable weight  $w$ . The set  $\mathfrak{U}_w$  contains only one element if and only if

$$\text{TM} \subseteq \text{WCh}. \quad (5.4)$$

*Proof.* Let (5.4) hold. In order to prove the uniqueness of extension of  $w$  it is sufficient to establish the equality

$$\rho(u, v) = \rho_w(u, v) \quad (5.5)$$

for every pair of distinct nonadjacent vertices  $u, v$  and every  $\rho \in \mathfrak{U}_w$ . Let  $u, v$  be distinct nonadjacent vertices for which  $\{u, v\} \notin \text{TM}$ . In this case, arguing as in the verification of correctness of the definition (5.2), we see that the value  $\rho(u, v)$  does not depend on the choice of  $\rho \in \mathfrak{U}_w$ . Hence, (5.5) holds. Consider now the case when  $\{u, v\} \in \text{TM}$ . In view of inclusion (5.4) the vertices are well chained. This and the definition of the subdominant pseudoultrametric  $\rho_w$  imply

$$\rho_w(u, v) = 0. \quad (5.6)$$

Since  $G$  is a connected graph, by Theorem 3.7 we obtain  $\rho(u, v) \leq \rho_w(u, v)$ . Moreover,  $0 \leq \rho(u, v)$  for  $u, v \in V(G)$ . The past two inequalities and (5.6) imply (5.5).

Let (5.4) be false. Let us prove the nonuniqueness of the extensions of  $w$ .

Let  $\{u_0, v_0\} \in \text{TM} \setminus \text{WCh}$ . We have  $\rho_w(u_0, v_0) > 0$  because  $\{u_0, v_0\} \notin \text{WCh}$ , so that there exists  $\varepsilon_0 > 0$  for which

$$\max_{e \in E(P)} w(e) \geq \varepsilon_0 \quad (5.7)$$

for every  $P \in \mathfrak{P}_{u,v}$ . Let  $\varepsilon_1$  be  $\varepsilon_2$  two distinct numbers from  $[0, \varepsilon_0)$ . Consider the graph  $\tilde{G}$  with

$$V(\tilde{G}) = V(G) \text{ и } E(\tilde{G}) = E(G) \cup \{\{u_0, v_0\}\},$$

i.e.,  $\tilde{G}$  can be obtained from  $G$  by adding the edge  $\{u_0, v_0\}$ . We define on  $\tilde{G}$  the weights  $w_i : E(\tilde{G}) \rightarrow \mathbb{R}^+$ ,  $i = 1, 2$  by the rule

$$w_i(e) = \begin{cases} w(e) & \text{if } e \in E(G) \\ \varepsilon_i & \text{if } e = \{u_0, v_0\}. \end{cases}$$

Let us verify that the weight  $w_1$  is pseudoultrametrizable. In accordance with Theorem 3.3 it is sufficient to show that for every cycle  $C \subseteq \tilde{G}$  there are two distinct edges  $e_1, e_2 \in E(C)$  satisfying

$$\max_{e \in E(C)} w_1(e) = w_1(e_1) = w_1(e_2). \quad (5.8)$$

Let  $C \subseteq \tilde{G}$ . If  $\{u_0, v_0\} \notin E(C)$ , then the existence of such  $e_1, e_2$  follows from the pseudoultrametrizability of  $w$ . Suppose that  $\{u_0, v_0\} \in E(C)$ . We can get a path  $P \in \mathfrak{P}_{u_0, v_0}$  by removing the edge  $\{u_0, v_0\}$  from the cycle  $C$ . Since  $\{u_0, v_0\} \in \text{TM}$ , there are two distinct edges  $e_1, e_2 \in E(P)$  such that

$$\max_{e \in E(P)} w_1(e) = \max_{e \in E(P)} w(e) = w(e_1) = w(e_2) = w_1(e_1) = w_1(e_2).$$

In virtue of inequalities (5.7) and  $\varepsilon_1 < \varepsilon_0$  we see that (5.8) holds for these edges. Consequently, the weight  $w_1 : E(\tilde{G}) \rightarrow \mathbb{R}^+$  is pseudoultrametrizable. In the same way we can verify that  $w_2 : E(\tilde{G}) \rightarrow \mathbb{R}^+$  is also a pseudoultrametrizable weight. Let  $\rho_1$  and  $\rho_2$  be two pseudoultrametrics on  $V(\tilde{G}) = V(G)$  extending  $w_1$  and, respectively,  $w_2$ . Then it is clear that  $\rho_1$  and  $\rho_2$  extend also the weight  $w$ . Since  $\varepsilon_1 \neq \varepsilon_2$ , we have  $w_1 \neq w_2$ . Thus  $\rho_1 \neq \rho_2$ . Consequently the extension of the weight  $w$  to pseudoultrametrics is not unique if condition (5.4) is violated.  $\square$

**Remark 5.8.** If  $G$  is a nonempty disconnected graph, then, for every pseudoultrametrizable weight  $w$ , the inequality

$$\text{card}(\mathfrak{U}_w) \geq \mathfrak{c}$$

holds where  $\mathfrak{c}$ , as usual, is the cardinality of continuum. The last inequality immediately follows from (3.3).

**Example 5.9.** For complete  $k$ -partite graphs with  $k \geq 2$  the uniqueness of extension of  $w$  is equivalent to the equality

$$\rho_w = \rho_{0,w}, \quad (5.9)$$

where  $\rho_w$  is the subdominant pseudoultrametric, and  $\rho_{0,w}$  is defined by equality (5.2). Indeed, every complete  $k$ -partite graph is connected for  $k \geq 2$ . In correspondence with theorems 3.3 and 5.2 the pseudoultrametric  $\rho_w$  is the greatest element of  $(\mathfrak{U}_w, \preceq)$  and  $\rho_{0,w}$  is the least element of this poset. Consequently, for every  $\rho \in \mathfrak{U}_w$ , we have

$$\rho_{0,w} \preceq \rho \preceq \rho_w,$$

and by (5.9)

$$\rho_{0,w} = \rho = \rho_w.$$

Let us show also that equality (5.9) is equivalent to inclusion (5.4). Note that in the proof of Theorem 5.2 the equality

$$\rho_{0,w}(u, v) = \rho_w(u, v) \quad (5.10)$$

was established for  $\rho_{0,w}(u, v) > 0$ . Moreover, (5.10) must hold for adjacent  $u$  and  $v$  because  $\rho_{0,w}, \rho_w \in \mathfrak{U}_w$ . Directly from the definitions we have

$$\text{WCh} = \{\{u, v\} : u \text{ and } v \text{ are non-adjacent, } u \neq v \text{ and } \rho_w(u, v) = 0\}, \quad (5.11)$$

and, for complete  $k$ -partite graphs with  $k \geq 2$ ,

$$\text{TM} = \{\{u, v\} : u \text{ and } v \text{ are non-adjacent, } u \neq v \text{ and } \rho_{0,w}(u, v) = 0\}.$$

Thus, (5.9) is equivalent to the equality  $\text{WCh} = \text{TM}$ . By  $\rho_{0,w} \preceq \rho_w$ , we have the inclusion  $\text{TM} \supseteq \text{WCh}$ . Consequently (5.9) is equivalent to the converse inclusion  $\text{TM} \subseteq \text{WCh}$ .

**Example 5.10.** Let  $G$  be a tree and  $w : E(G) \rightarrow \mathbb{R}^+$  be a strictly positive weight. The weight  $w$  has the unique extension if and only if  $\text{TM} = \emptyset$ .

Indeed, every two vertices of the tree  $G$  are connected by the unique path. From this uniqueness, the strict positiveness of  $w$ , formula (5.11) and the definition of  $\rho_w$  it follows that  $\text{WCh} = \emptyset$ . By Theorem 5.7, the last equality implies that the uniqueness of extension of  $w$  is equivalent to  $\text{TM} = \emptyset$ .

**Remark 5.11.** The equality  $\text{TM} = \emptyset$  is equivalent to the fact that the problem of extending of strictly positive pseudoultrametrizable weight  $w$  to a pseudoultrametric has the unique solution for every  $G$  satisfying condition (ii) of Theorem 4.4. For trees this equality is equivalent to the following statement.

If  $e_1$  and  $e_2$  are two distinct edges of the tree  $G$  such that  $w(e_1) = w(e_2)$ , then for every path  $P \subseteq G$  including  $e_1$  and  $e_2$  there exists an edge  $e_3 \in E(P)$  such that  $w(e_3) > w(e_1)$ .

## References

- [1] *J. A. Bondy, U.S.R. Murty*, Graph Theory. Graduate Texts in Mathematics. New York, Springer, **244**, 2008.
- [2] *O. Dovgoshey, O. Martio, M. Vuorinen*, Metrization of weighted graphs // Reports in Math. University of Helsinki, **516**, 2011.
- [3] *N. Jardine and R. Sibson*, Mathematical Taxonomy. Part II. London, Willey, 1971.
- [4] *J.M. Bayod, J. Martinier-Maurica*, Subdominant ultrametrics// Proc. of AMS., **109**:3, 1990, 829-834.
- [5] *J.P.J. Jardine, N. Jardine, R. Sibson*, The structure and construction of taxonomic hierarchies // Math. Biosciences, **1**, 1967, 171-179.
- [6] *R. Rammal, J.C. Angles d'Auriac, and B. Doucot*, On degree of ultrametricity // J. Physique Lett., **46**, 1985, 945-952.
- [7] *R. Rammal, G. Toulouse, and M. A. Virasoro*, Ultrametricity for physicist// Rev. Modern Physics, **58**:3, 1986, 765-788.
- [8] *D. Marcu*, A study on metric and statistical analysis // Studia Univ. "Babes-Bolyai", Mathematica, **49**:3, 2004, 43-74.
- [9] *O. Dovgoshey and O. Martio*, Blow up of balls and coverings in metric spaces // Manuscripta Math., **127**, 2008, 89-120.
- [10] *O. Dovgoshey, D. Dordovskyi*, Ultrametricity and Metric Betweenness in Tangent Spaces to Metric Spaces // P-adic Numbers, Ultrametric Analysis and Applications, **2**:2, 2010, 100-113.
- [11] *Alex J. Lemin*, On ultrametrization of general metric spaces // Proc. of AMS, **131**:3, 2002, 979-989.
- [12] *D. Dordovskyi, O. Dovgoshey and E. Petrov*, Diameter and Diametrical Pairs of points in Ultrametric Spaces // P-adic Numbers, Ultrametric Analysis and Applications, **3**:4, 2011, 253-262.
- [13] *Sam B. Nadler, Jr.* Continuum Theory. An Introduction. New York, Marcel Dekker, INC, 1992.
- [14] *Anthony G. O'Farrel* When Uniformly-continuous Implies Bounded // Irish Math. Soc. Bulletin, **53**, 2004, 53-56.

**Oleksiy Dovgoshey**

Institute of Applied Mathematics and Mechanics of NASU, R. Luxemburg str. 74,  
Donetsk 83114, Ukraine

**E-mail:** aleksdov@mail.ru

**Evgeniy Petrov**

Institute of Applied Mathematics and Mechanics of NASU, R. Luxemburg str. 74,  
Donetsk 83114, Ukraine

**E-mail:** eugeniy.petrov@gmail.com